Exam 6: Ch. 10.1-10.4, 10.7, 11.1-11.3, 11.5, 11.6 **NAME** Answer Key
Math 176, Precalculus, Section 6265 *NO TI-89 CALCULATORS!!!!!!!!*
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100 points. Show all work and steps taken to arrive at the solution to receive full credit. You may use a calculator. CHECK YOUR WORK!!!!

1. (18 points) Graph each of the following conic sections and find the desired information.

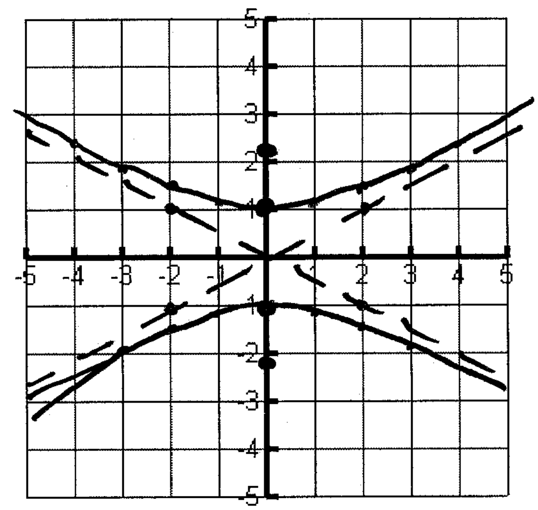
A. $y^2 - \frac{x^2}{4} = 1$ HYPERBOLA - UP/DOWN

CTR: $(0,0)$

$a^2 = 1$ $b^2 = 4$ $b = 2$

$c^2 = a^2 + b^2$ $c = \sqrt{5}$

VERTICES: $(0,1)$ $(0,-1)$



Foci: $(0,\sqrt{5})$ $(0,-\sqrt{5})$

Asymptotes: $y = \frac{1}{2}x$, $y = -\frac{1}{2}x$

B. $\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$ ELLIPSE

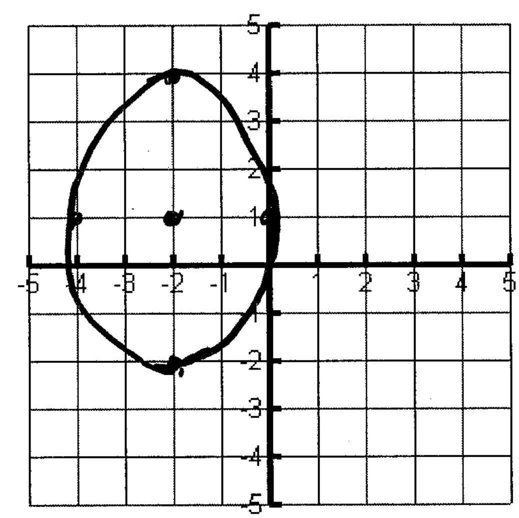
$a^2 = 9$ $a = 3$

$b^2 = 4$ $b = 2$

$c^2 = a^2 - b^2 = 5$

$c = \sqrt{5}$

$e = \frac{c}{a} = \frac{\sqrt{5}}{3}$



Center: $(-2,1)$

Eccentricity: $\sqrt{5}/3$

2. (8 pts) Find the focus and directrix of the parabola $y^2 = -8x$, Graph the parabola. Does it open up, down, left or right?

Focus $(-2,0)$

Directrix $X = 2$

Opens LEFT

$y^2 = 4PX$ $\qquad$ $4P = -8$

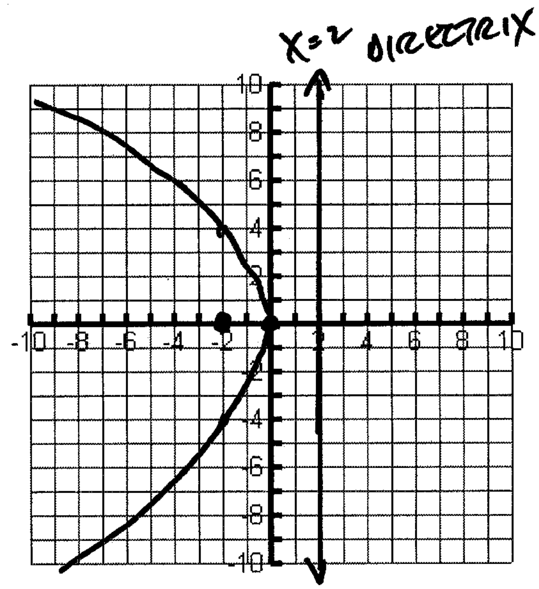
$P = -2$

Vertex $= (0,0)$

Focus: $(-2,0)$

Directrix $\quad X = -P$

$\boxed{X = 2}$



3. (6 pts) Find the **directrix** and an **equation** for the parabola with vertex $(0,0)$ and focus $(-3,0)$.

$y^2 = 4PX$ $\qquad$ $p = -3$

$y^2 = 4(3)X$

$y^2 = -12X$

Directrix $X = 3$

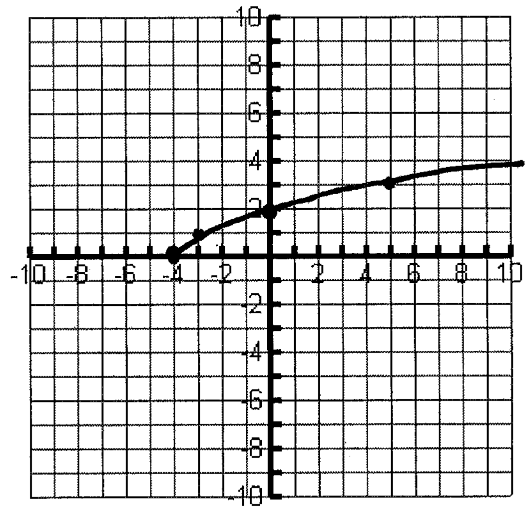
Equation $y^2 = -12X$

$X+4$

4. (8 pts) Consider the parametric equations $\begin{matrix} x = t - 4 \\ y = \sqrt{t} \end{matrix}$, $t \geq 0$.

A. Sketch the parametric curve.

t	x	y
0	-4	0
1	-3	1
2	-2	$\sqrt{2}$
3	-1	$\sqrt{3}$
4	0	2
9	5	3
16	12	4



B. Find a rectangular-coordinate equation for the curve by eliminating the parameter, t.

$x = t - 4$

$y = \sqrt{t}$

$y^2 = t$

$x = y^2 - 4$ $\quad\quad$ $x + 4 = y^2$

5. (6 pts) Given the sequence $5, -15, 45, -135, 405, \ldots$

A. Find the formula for the nth term, a_n

$r = \frac{-15}{3} = -3$

$= \frac{45}{-15} = -3$

$a_n = a_1 r^{n-1}$

$\boxed{a_n = 5(-3)^{n-1}}$

B. Find the sum of the first 10 terms, S_{10}

$S_{10} = a_1 \frac{(1 - r^n)}{(1 - r)} = \frac{5(1 - (-3)^{10})}{1 - (-3)} = \boxed{-73810}$

6. *(2 pts each)* TRUE or FALSE. Determine whether each statement is True or False

a. If the eccentricity of an ellipse is near 0, then the ellipse is close to circular in shape. TRUE

b. If the focus of a parabola is very close to its vertex, then the parabola is very narrow. TRUE

c. If $|r| > 1$, then the infinite geometric series has a sum $S = \dfrac{a}{1-r}$. FALSE

d. It is not necessary to show that $P(1)$ is true for a mathematical induction proof. FALSE

e. In a binomial expansion, the binomial coefficient is defined as $n\text{Pr} = \dfrac{n!}{(n-r)!}$ FALSE

7. (16 pts) Complete the square to determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, vertices, and length of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation.

$$x^2 - 4x - 4y^2 - 24y - 48 = 0$$

$x^2 - 4x \quad -4y^2 - 24y = 48$

$x^2 - 4x + 4 \quad -4(y^2 + 6y + 9) = 48 + 4 - 36 = 16$

$\dfrac{(x-2)^2}{16} - \dfrac{4(y+3)^2}{16} = \dfrac{16}{16}$

$\dfrac{(x-2)^2}{16} - \dfrac{(y+3)^2}{4} = 1$

$c^2 = a^2 + b^2$

$= 16 + 4 = 20$

$c = 2\sqrt{5}$

$a^2 = 16 \quad a = 4 \quad b = 2$

Type of conic: HYPERBOLA

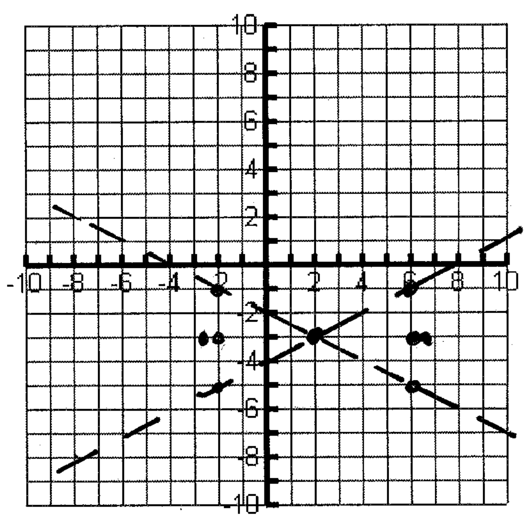
CTR: $(2, -3)$

OPENS RT / LEFT

VERTICES: $(-2, -3)$ $(6, -3)$

FOCI: $(2 - 2\sqrt{5}, -3)$ $(2 + 2\sqrt{5}, -3)$

$(-2.47, -3)$ $(6.47, -3$



ASYMPTOTES:

$y = \frac{1}{2}x + b$

$-3 = \frac{1}{2}(2) + b$

$-4 = b$

$y = -\frac{1}{2}x + b$

$-3 = -\frac{1}{2}(2) + b$

$-3 = -1 + b \quad b = -2$

$y = -\frac{1}{2}x - 2$

$y = \frac{1}{2}x - 4$

8. (6 pts) Find the sum. Show your work. $\sum_{i=1}^{8}\left[1+(-1)^i\right]$

$[1+(-1)^1] + [1+(-1)^2] + [1+(-1)^3] + [1+(-1)^4] + [1+(-1)^5]$
$+ [1+(-1)^6] + [1+(-1)^7] + [1+(-1)^8]$

$= 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2$

$= \boxed{8}$

9. (6 pts) Find the first five terms of the given recursively defined sequence:
$a_n = a_{n-1} + 2a_{n-2}$ and $a_1 = 4,\ a_2 = 5$

$a_1 = 4$

$a_2 = 5$

$a_3 = a_2 + 2(a_1)$
$= 5 + 2(4) = 13$

$a_4 = a_3 + 2(a_2) = 13 + 2(5) = 23$

$a_5 = a_4 + 2(a_3) = 23 + 2(13) = 23 + 26 = 49$

$\boxed{\{4, 5, 13, 23, 49\}}$

10. (8 pts) Use mathematical induction to prove that the formula is true for all natural numbers n.

$$1\cdot3+2\cdot4+3\cdot5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6}$$

$P(1) = \frac{1(2)(9)}{6} = 3 = 1\cdot 3$ ✓

$\underline{1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots + k(k+2)} + (k+1)(k+2+1) = \frac{(k+1)(k+2)(2(k+1)+7)}{6}$

$\frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) = \frac{(k+1)(k+2)(2k+9)}{6}$ RHS

$\frac{k(k+1)(2k+7) + 6(k^2+4k+3)}{6}$

$(k^2+3k+2)(2k+9)$

$2k^3 + 6k^2 + 4k + 9k^2 + 27k + 18$

$= \frac{k(2k^2+9k+7) + 6k^2+24k+18}{6}$

$= 2k^3+15k^2+31k+18$

$= 2k^3 + 9k^2 + 6k^2 + 7k + 24k + 18$

$= 2k^3 + 15k^2 + 31k + 18$

LHS = RHS ✓

11. (8 pts) Use the Binomial Theorem (or Pascal's Triangle) to expand $(2x+3y)^7$.

$(2x+3y)^7 = \binom{7}{0}(2x)^7 + \binom{7}{1}(2x)^6(3y) + \binom{7}{2}(2x)^5(3y)^2 + \binom{7}{3}(2x)^4(3y)^3$

$+ \binom{7}{4}(2x)^3(3y)^4 + \binom{7}{5}(2x)^2(3y)^5 + \binom{7}{6}(2x)(3y)^6 + \binom{7}{7}(3y)^7$

$= 128x^7 + 7(64x^6)(3y) + 21(32x^5)(9y^2) + 35(16x^4)(27y^3)$

$+ 35(8x^3)(81y^4) + 21(4x^2)(243y^5) + 7(2x)(729y^5)$

$+ 2167y^7$

$= \boxed{128x^7 + 1344x^6y + 6048x^5y^2 + 15120x^4y^3 + 22680x^3y^4 + 20412x^2y^5 + 10206xy^6 + 2167y^7}$

***BONUS** (total of **10** extra points)*

(5 pts) Find the last two terms in the expansion of $\left(a^{2/3} + a^{1/3}\right)^{25}$.

$$\binom{25}{24}\left(a^{2/3}\right)\left(a^{1/3}\right)^{24} + \binom{25}{25}\left(a^{1/3}\right)^{25}$$

$$25\left(a^{2/3}\right)\left(a^{24/3}\right) + 1\left(a^{25/3}\right)$$

$$\boxed{\begin{aligned} 25a^{26/3} + a^{25/3} &= a^{8}\left(25a^{2/3} + a^{1/3}\right) \\ &= a^{8}\left(25\sqrt[3]{a^{2}} + \sqrt[3]{a}\right) \end{aligned}}$$

(5 pts) Find the next 3 numbers in the sequence:

1, 5, 3, 7, 5, 9, 7, 11, 9, 13

+4 −2 +4 −2 +4 −2